Chapter One

Introduction

This book represents a modern treatment of classical control theory of continuous- and discrete-time linear systems. Classical control theory originated in the fifties and attained maturity in the sixties and seventies. During that time control theory and its applications were among the most challenging and interesting scientific and engineering areas. The success of the space program and the aircraft industry was heavily based on the power of classical control theory.

The rapid scientific development between 1960 and 1990 brought a tremendous number of new scientific results. Just within electrical engineering, we have witnessed the real explosion of the computer industry in the middle of the eighties, and the rapid development of signal processing, parallel computing, neural networks, and wireless communication theory and practice at the beginning of the nineties. In the years to come many scientific areas will evolve around vastly enhanced computers with the ability to solve by virtually brute force very complex problems, and many new scientific areas will open in that direction. The already established "information superhighway" is maybe just a synonym for the numerous possibilities for "informational breakthrough" in almost all scientific and engineering areas with the use of modern computers. Neural networks—dynamic systems able to process information through large number of inputs and outputs—will become specialized "dynamic" computers for solving specialized problems.

Where is the place of classical (and modern) control theory in contemporary scientific, industrial, and educational life? First of all, classical control theory values have to be preserved, properly placed, and incorporated into modern scientific knowledge of the nineties. Control theory will not get as much attention and

recognition as it used to enjoy in the past. However, control theory is concerned with dynamic systems, and dynamics is present, and will be increasingly present, in almost all scientific and engineering disciplines. Even computers connected into networks can be studied as dynamic systems. Communication networks have long been recognized as dynamic systems, but their models are too complex to be studied without the use of powerful computers. Traffic highways of the future are already the subject of broad scale research as dynamic systems and an intensive search for the best optimal control of networks of highways is underway. Robotics, aerospace, chemical, and automotive industries are producing every day new and challenging models of dynamic systems which have to be optimized and controlled. Thus, there is plenty of room for further development of control theory applications, both behind or together with the "informational power" of modern computers.

Control theory must preserve its old values and incorporate them into modern scientific trends, which will be based on the already developed fast and reliable packages for scientific numerical computations, symbolic computations, and computer graphics. One of them, MATLAB, is already gaining broad recognition from the scientific community and academia. It represents an expert system for many control/system oriented problems and it is widely used in industry and academia either to solve new problems or to demonstrate the present state of scientific knowledge in control theory and its applications. The MATLAB package will be extensively used throughout of this book to solve many control theory problems and allow deeper understanding and analysis of problems that would not otherwise be solvable using only pen and paper.

Most contemporary control textbooks originated in the sixties or have kept the structure of the textbooks written in the sixties with a lot of emphasis on frequency domain techniques and a strong distinction between continuous- and discrete-time domains. At the present time, all undergraduate students in electrical engineering are exposed to discrete-time systems in their junior year while studying linear systems and signals and digital signal processing courses so that parallel treatment of continuous- and discrete-time systems saves time and space. The time domain techniques for system/control analysis and design are computationally more powerful than the frequency domain techniques. The time domain techniques are heavily based on differential/difference equations and linear algebra, which are very well developed areas of applied mathematics, for which efficient numerical methods and computer packages exist. In addition, the

state space time domain method, to be presented in Chapter 3, is much more convenient for describing and studying higher-order systems than the frequency domain method. Modern scientific problems to be addressed in the future will very often be of high dimensions.

In this book, the reader will find parallel treatment of continuous- and discrete-time systems with *emphasis on continuous-time control systems and on time domain techniques (state space method)* for analysis and design of linear control systems. However, all fundamental concepts known from the frequency domain approach will be presented in the book. Our goal is to present the essence, the fundamental concepts, of classic control theory—something that will be valuable and applicable for modern dynamic control systems.

The reader will find that some control concepts and techniques for discrete-time control systems are not fully explained in this book. The main reason for this omission is that those "untreated topics" can be simply obtained by extending the presented concepts and techniques given in detail for continuous-time control systems. Readers particularly interested in discrete-time control systems are referred to the specialized books on that topic (e.g. Ogata, 1987; Franklin *et al.*, 1990; Kuo, 1992; Phillips and Nagle, 1995). Instructors who are not enthusiastic about the simultaneous presentation of both continuous- and discrete-time control systems can completely omit the "discrete-time parts" of this book and give only continuous-time treatment of control systems. This book contains an introduction to discrete-time systems that naturally follows from their continuous-time counterparts, which historically are first considered, and which physically represent models of real-world systems.

Having in mind that this textbook will be used at a time when control theory is not at its peak, and is merging with other scientific fields dealing with dynamic systems, we have divided this book into two independent parts. In Chapters 2–5 we present *fundamental control theory methods and concepts*: transfer function method, state space method, system controllability and observability concepts, and system stability. In the next four chapters, we mostly deal with *applications* so that techniques useful for *design* of control systems are considered. In Chapter 10, an overview of modern control areas is given. A description of the topics considered in the introductory chapter of this book is given in the next paragraph.

Chapter Objectives

In the first chapter of this book, we introduce continuous- and discrete-time invariant linear control systems, and indicate the difference between open-loop

and closed-loop (feedback) control. The two main techniques in control system analysis and design, i.e. state space and transfer function methods, are briefly discussed. Modeling of dynamic systems and linearization of nonlinear control systems are presented in detail. A few real-world control systems are given in order to demonstrate the system modeling and linearization. Several other models of real-world dynamic control systems will be considered in the following chapters of this book. In the concluding sections, we outline the book's structure and organization, and indicate the use of MATLAB and its CONTROL and SIMULINK toolboxes as teaching tools in computer control system analysis and design.

1.1 Continuous and Discrete Control Systems

Real-world systems are either static or dynamic. Static systems are represented by algebraic equations, and since not too many real physical systems are static they are of no interest to control engineers. Dynamic systems are described either by differential/difference equations (also known as *systems with concentrated* or *lumped parameters*) or by partial differential equations (known as *systems with distributed parameters*). Distributed parameter control systems are very hard to study from the control theory point of view since their analysis is based on very advanced mathematics, and hence will not be considered in this book. At some schools distributed parameter control systems are taught as advanced graduate courses. Thus, we will pay attention to concentrated parameter control systems, i.e. dynamic systems described by differential/difference equations. It is important to point out that many real physical systems belong to the category of concentrated parameter control systems and a large number of them will be encountered in this book.

Consider, for example, dynamic systems represented by scalar differential/difference equations

$$\dot{x}(t) = f_c(x(t)), \qquad x(t_0) = x_0 \tag{1.1}$$

$$x(k+1) = f_d(x(k)), \qquad x(k_0) = x_0 \tag{1.2}$$

where t stands for continuous-time, k represents discrete-time, subscript c indicates continuous-time and subscript d is used for discrete-time functions. By solving these equations we learn about the system's evolution in time (system

response). If the system is under the influence of some external forces, or if we intend to change the system response intentionally by adding some external forces, then the corresponding system is represented by the so-called controlled differential/difference equation, that is

$$\dot{x}(t) = f_c(x(t), u(t)), \qquad x(t_0) = x_0 \tag{1.3}$$

$$x(k+1) = f_d(x(k), u(k)), \qquad x(k_0) = x_0 \tag{1.4}$$

where $u(t)$ and $u(k)$ play the role of control variables. By changing the control variable we hope that the system behavior can be changed in the desired direction, in other words, we intend to use the control variable such that the system response has the desired specifications. When we are able to achieve this goal, we are actually controlling the system behavior.

The general control problem can be formulated as follows: *find the control variable such that the solution of a controlled differential/difference equation has some prespecified characteristics.* This is a quite general definition. In order to be more specific, we have to precisely define the class of systems for which we are be able to solve the general control problem. Note that the differential/difference equations defined in (1.1)–(1.4) are nonlinear. In general, it is hard to deal with nonlinear systems. Nonlinear control systems are studied at the graduate level. In this undergraduate control course, we will study only *linear time invariant control systems*.

Continuous- and discrete-time *linear time invariant dynamic systems* are described, respectively, by linear differential and difference equations with constant coefficients. Mathematical models of such systems having one input and one output are given by

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = u(t) \tag{1.5}$$

and

$$y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1 y(k+1) + a_0 y(k) = u(k) \tag{1.6}$$

where n is the order of the system, y is the *system output* and u is the external forcing function representing the *system input*. In addition to the *external forcing function* the system is also driven by its *internal forces* coming from the *system*

initial conditions. For continuous-time systems, the initial conditions are specified by known values of the system output derivatives up to the order of $n-1$ at the initial time t_0, that is

$$y(t_0),\ \frac{dy(t_0)}{dt},\ ...,\ \frac{d^{n-1}y(t_0)}{dt^{n-1}} \tag{1.7}$$

In the discrete-time domain the initial conditions are specified by

$$y(k_0),\ y(k_0+1),\ ...,\ y(k_0+n-1) \tag{1.8}$$

It is interesting to point out that in the discrete-time domain the initial conditions carry information about the evolution of the system output in time from k_0 to k_0+n-1. In this book, we study only time invariant continuous and discrete systems for which the coefficients $a_i, i=0,1,...,n-1$, are constants. A block diagram representation of such a system is given in Figure 1.1.

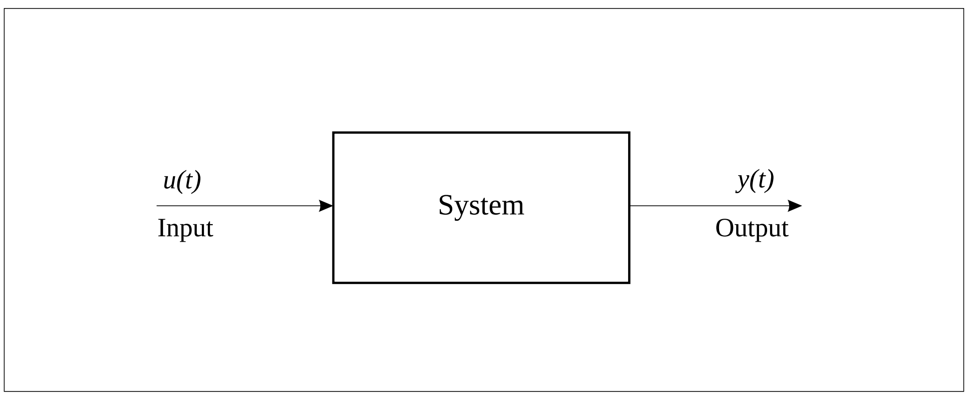

Figure 1.1: Input–output block diagram of a system

In general, the input function can be differentiated by the system so that the more general descriptions of time invariant continuous and discrete systems are given by

$$\begin{aligned} &\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) \\ &\quad = b_m\frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}} + \cdots + b_1\frac{du(t)}{dt} + b_0 u(t) \end{aligned} \tag{1.9}$$

and

$$y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1 y(k+1) + a_0 y(k)$$
$$= b_m u(k+m) + b_{m-1}u(k+m-1) + \cdots + b_1 u(k+1) + b_0 u(k) \tag{1.10}$$

where all coefficients $a_i, i = 1, 2, ..., n$, and $b_j, j = 0, 1, ..., m$, are constant.

The problem of obtaining differential (difference) equations that describe dynamics of real physical systems is known as *mathematical modeling*. In Sections 1.4 and 1.5 this problem will be addressed in detail and mathematical models for several real physical systems will be derived.

Basic Linear Control Problem

In summary, we outline the basic problem of the linear control theory. The problem of finding the system response for the given input function $u(t)$ or $u(k)$ is basically the straightforward problem of solving the corresponding linear differential or difference equation, (1.9) or (1.10). This problem can be solved by using standard knowledge from mathematical theory of linear differential and/or difference equations. However, *the linear control problem is much more challenging, namely the input function $u(t)$ has to be found such that the linear system response has the desired behavior*. A simplified version of the above basic linear control problem will be defined in Chapter 6 using the notion of system feedback. The basic linear control problem can be studied either in the time domain (state space approach) or in the frequency domain (transfer function approach). These two approaches will be presented in Chapters 2 and 3.

1.2 Open-Loop and Closed-Loop Control Systems

It seems that if an input function can be found such that the corresponding system has the desired response, then the control problem of interest is solved. This is true, but is it all that we need? Assume that $u(t)$ is such a function, which is apparently a function of time. Imagine that due to parameter variations or due to aging of the system components the system model is a little bit different than the original one or even worse that the coefficients $a_i, i = 0, 1, 2, ..., n-1$; $b_j, j = 0, 1, 2, ..., m$, in equation (1.9) are not very precisely known. Then the function $u(t)$, given as a precomputed time function, might not produce a satisfactory solution (especially in the long run). One may try to solve the problem again and get a new expression for $u(t)$ at the expense of additional

computation, which is fine if the system parameters are exactly known, but this approach will not bring any improvement in the case when the system coefficients, obtained either analytically or experimentally, are known only with certain accuracy. This *precomputed time function* $u(t)$ (or $u(k)$ in the discrete-time domain), which solves the control problem, is known as the *open-loop control*.

Imagine now that one is able, in an attempt to solve the basic linear control problem, to obtain the desired input function as a function of the system desired response, or even more precisely as a function of some essential system variables that completely determine the system dynamics. These essential system variables are called the *state space variables*. It is very natural to assume that for a system of order n, a collection of n such variables exist. Denote the state space variables by $x_1(t), x_2(t), ..., x_n(t)$. These state variables at any given time represent the actual state of the system. Even if some parameters of the system are changing in time or even if some coefficients in (1.9) are not precisely known, the state variables $x_1(t), x_2(t), ..., x_n(t)$ will reflect exactly the state of the system at any given time. The question now is: can we get the required control variable (system input) as a function of the state space variables? If the answer is yes, then the existence of such a $u(\mathbf{x}(t))$ indicates the existence of the so-called *state feedback control*. In some cases it is impossible to find the feedback control, but for the linear time invariant systems studied in this book, linear feedback control always exists. The linear feedback control is a linear function of the state space variables, that is

$$\begin{aligned} u(\mathbf{x}) &= F_1 x_1 + F_2 x_2 + \cdots + F_n x_n = \mathbf{F}\mathbf{x} \\ \mathbf{F} &= [F_1, F_2, ..., F_n] \\ \mathbf{x} &= [x_1, x_2, ..., x_n]^T \end{aligned} \tag{1.11}$$

where the coefficient matrix $\mathbf{F}$ is the *feedback gain* and the vector $\mathbf{x}$ is known as the *state space vector*. It is sometimes desirable (and possible) to achieve the goal by using instead of the state feedback control $u(\mathbf{x}(t))$, the so-called *output feedback control* given by $u(y(t)) = u(y(\mathbf{x}(t)))$, which in general does not contain all state variables, but only a certain collection of them. In Section 3.1 we will learn how to relate the output and state space variables.

Open-loop control, state feedback control, and output feedback control are represented schematically in Figure 1.2.

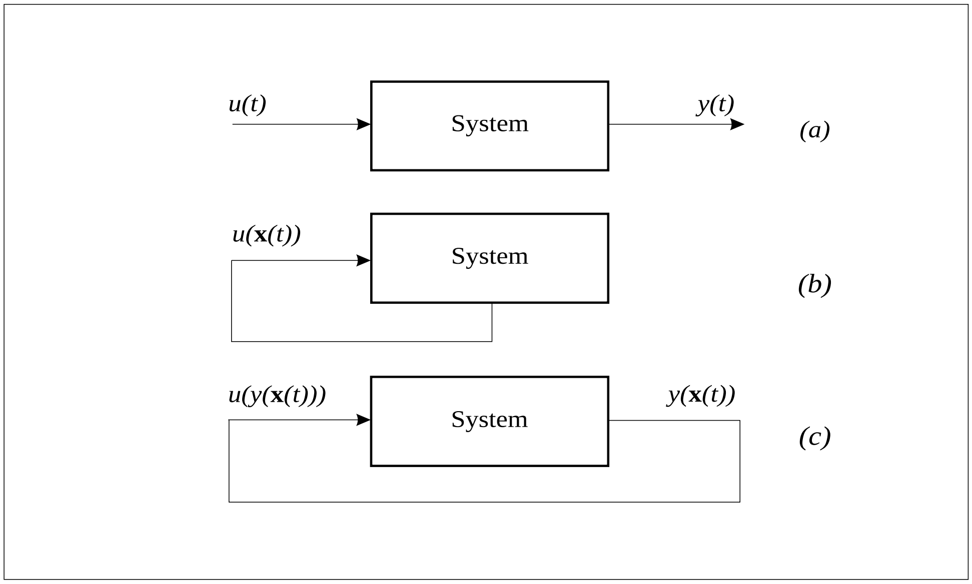

Figure 1.2: Open-loop (a), state feedback (b), and output feedback (c) controls

The system represented in Figure 1.1 and given in formulas (1.5)–(1.10) has only one input u and one output y. Such systems are known as *single-input single-output systems*. In general, systems have several inputs and several outputs, say r inputs and p outputs. In that case, we have

$$\mathbf{u} = [u_1, u_2, ..., u_r]^T, \quad \mathbf{y} = [y_1, y_2, ..., y_p]^T \tag{1.12}$$

and the matrix $\mathbf{F}$ is of dimension $r \times n$. These systems are known as *multi-input multi-output systems*. They are also called *multivariable control systems*. A block diagram for a multi-input multi-output system is represented in Figure 1.3.

Feedback control is almost always desirable as a solution to the general control problem, and only in rare cases and in the cases when it is impossible to find the feedback control has one to stick with open-loop control. Throughout of book we will see and discuss many advantages of feedback control. *The main role of feedback is to stabilize the system under consideration*. The feedback

also reduces the effect of uncertainties in the system model. In addition, it efficiently handles system parameter changes and external disturbances attacking the system, by simply reducing the system output sensitivity to all of these undesired phenomena. Most of these good feedback features will be analytically justified in the follow-up chapters.

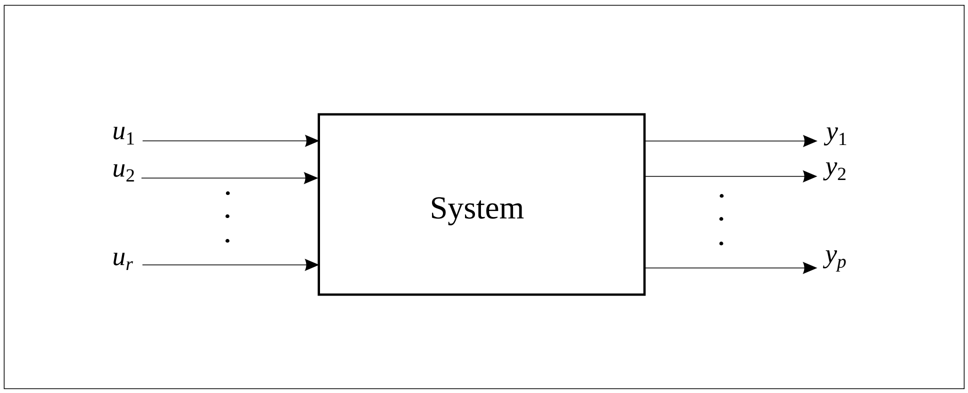

Figure 1.3: Block diagram of a multi-input multi-output system

However, one has to be careful when applying feedback since it changes the structure of the original system. Consider the system described by equation (1.3) under the state feedback control, that is

$$\dot{x}(t) = f_c(x(t), u(x(t))) = \mathcal{F}_c(x(t)), \qquad x(t_0) = x_0 \tag{1.13}$$

Thus, after the feedback is applied, a new dynamic system is obtained, in other words, for different values of $u(x(t))$ we will get different expressions for the right-hand side of (1.13) so that equation (1.13) will represent different dynamic systems.

1.3 State Space and Transfer Functions

In analysis and design of linear time invariant systems two major approaches are available: the time domain state space approach and the frequency domain transfer function approach. Both approaches will be considered in detail in Chapters 2 and 3. Electrical engineering students are partially familiar with

these methods and the corresponding terminology from undergraduate courses on linear systems and digital signal processing.

In this section only the main definitions are stated. This is necessary to maintain the continuum of presentation and to get mathematical models of some real physical systems in state space form. In addition, by being familiar with the notion of the state space, we will be able to present the linearization of nonlinear systems in the most general form.

In the previous section it is indicated that for an nth order system there are n essential state variables, the so-called state space variables, which form the corresponding state space vector $\mathbf{x}$. For linear time invariant systems, the vector of state space variables satisfies the linear time invariant vector differential (difference) equation known as the state space form. The state space equations of linear systems are defined by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_o \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{1.14}$$

and

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k), \quad \mathbf{x}(0) = \mathbf{x}_o \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) \end{aligned} \tag{1.15}$$

In view of the discussion in the previous section, the introduced constant matrices have the dimensions $\mathbf{A}^{n\times n}, \mathbf{B}^{n\times r}, \mathbf{C}^{p\times n}, \mathbf{D}^{p\times r}$. Of course, the given matrices for continuous and discrete systems have different entries. In formulas (1.14) and (1.15) the same notation is kept for both continuous and discrete systems for the sake of simplicity. In the case when a discrete-time system is obtained by sampling a continuous-time system, we will emphasize the corresponding difference by adding a subscript d for the discrete-time quantities, e.g. $\mathbf{A}_d, \mathbf{B}_d, \mathbf{C}_d, \mathbf{D}_d$.

The *system transfer function* for a single-input single-output time invariant continuous system is defined as *the ratio of the Laplace transform of the system output over the Laplace transform of the system input assuming that all initial conditions are zero*. This definition implies that the transfer function corresponding to (1.9) is

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{1.16}$$

Polynomial exponents in transfer functions of real physical systems always satisfy $n \geq m$. In the discrete-time domain, the $\mathcal{Z}$-transform takes the role of the Laplace

transform so that the discrete-time transform function is given by

$$G(z) = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} \tag{1.17}$$

For multi-input multi-output systems with r inputs and p outputs, transfer functions are matrices of order $p \times r$ whose entries are the corresponding transfer functions from the ith system input to the jth system output, $i = 1, 2, ..., r$; $j = 1, 2, ..., p$, that is

$$\mathbf{G}(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \dots & \dots & G_{1r}(s) \\ G_{21}(s) & \dots & \dots & \dots & \dots \\ \dots & \dots & G_{ji}(s) & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ G_{p1}(s) & \dots & \dots & \dots & G_{pr}(s) \end{bmatrix}^{p \times r} \tag{1.18}$$

Recall from basic circuits courses that while finding $G_{ji}(s) = Y_j(s)/U_i(s)$ *all other system inputs except for* $U_i(s)$ *must be set to zero*. Similarly, the discrete-time transfer function of a multi-input multi-output, time invariant, system is given by

$$\mathbf{G}(z) = \begin{bmatrix} G_{11}(z) & G_{12}(z) & \dots & \dots & G_{1r}(z) \\ G_{21}(z) & \dots & \dots & \dots & \dots \\ \dots & \dots & G_{ji}(z) & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ G_{p1}(z) & \dots & \dots & \dots & G_{pr}(z) \end{bmatrix}^{p \times r} \tag{1.19}$$

More will be said about the transfer function of multi-input multi-output systems in Chapter 2.

Since each entry in the matrix transfer functions given in (1.18) and (1.19) is a ratio of two polynomials with complex numbers, it is obvious that for higher-order systems the required calculations in the frequency domain are mathematically very involved so that the state space system representation is simpler than the corresponding frequency domain representation. In addition, since the state space method is based on linear algebra, for which numerous efficient mathematical methods have already been developed, the state space method is also more convenient from the computational point of view than the frequency domain method. However, the importance of the frequency domain representation lies in the simplicity of presenting some basic concepts, and hence it very often gives a better understanding of the actual physical phenomena occurring within the system.

1.4 Mathematical Modeling of Real Physical Systems

Mathematical modeling of real-world physical systems is based on the application of known physical laws to the given systems, which leads to mathematical equations describing the behavior of systems under consideration. The equations obtained are either algebraic, ordinary differential or partial differential. Systems described by algebraic equations are of no interest for this course since they represent static phenomena. Dynamic systems mathematically described by partial differential equations are known as systems with distributed parameters. The study of distributed parameter systems is beyond the scope of this book. Thus, we will consider only systems described by ordinary differential equations. These systems are also known as systems with lumped (concentrated) parameters. Even the lumped parameter systems are, in general, too difficult from the point of view of solving the general control problem described in Section 1.1, so that we have to limit our attention to lumped parameter systems described by linear time invariant differential equations. Fortunately, many control systems do have this form. Even more, control systems described by nonlinear differential equations can very often be linearized in the neighborhood of their nominal (operating) trajectories and controls assuming that these quantities are known, which is very often the case, so that nonlinear systems can be studied as linear ones. The linearization procedure will be independently considered in Section 1.6.

In the following the modeling procedure is demonstrated on a simple RLC electrical network given in Figure 1.4. Assume that the initial values for the

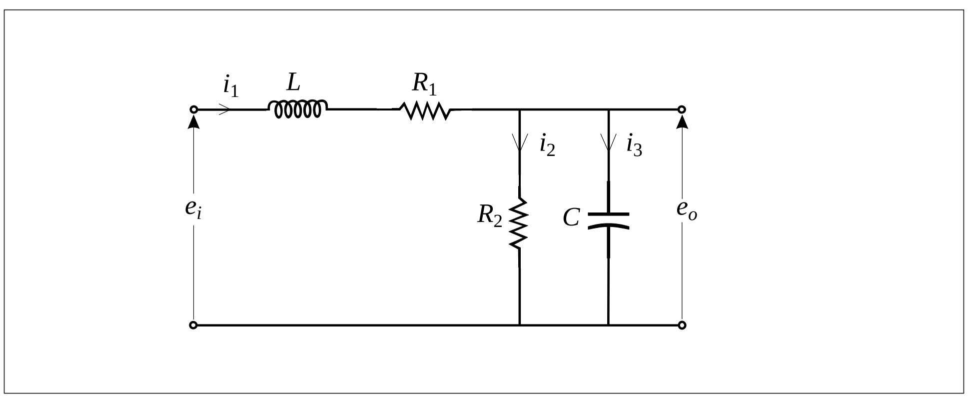

Figure 1.4: An RLC network

inductor current and capacitor voltage are zero. Applying the basic circuit laws (Thomas and Rosa, 1994) for voltages and currents, we get

$$e_i(t) = L\frac{di_1(t)}{dt} + R_1 i_1(t) + e_0(t) \tag{1.20}$$

$$e_0(t) = R_2 i_2(t) = \frac{1}{C}\int_0^t i_3(\tau)d\tau \quad \Rightarrow \quad i_3 = C\frac{de_0(t)}{dt} \tag{1.21}$$

$$i_1(t) = i_2(t) + i_3(t) \tag{1.22}$$

Using (1.21) in (1.22) produces

$$i_1(t) = \frac{1}{R_2}e_0(t) + C\frac{de_0(t)}{dt} \tag{1.23}$$

Taking the derivative of (1.23) and combining (1.20) and (1.23), we obtain the desired second-order differential equation, which relates the input and output of the system, and represents a mathematical model of the circuit given in Figure 1.4

$$\frac{d^2e_0(t)}{dt^2} + \left(\frac{L + R_1R_2C}{R_2LC}\right)\frac{de_0(t)}{dt} + \left(\frac{R_1 + R_2}{R_2LC}\right)e_0(t) = \frac{1}{LC}e_i(t) \tag{1.24}$$

Note that in this mathematical model $e_i(t)$ represents the system input and $e_0(t)$ is the system output. However, any of the currents and any of the voltages can play the roles of either input or output variables.

Introducing the following change of variables

$$\begin{gathered} x_1(t) = e_o \quad \Rightarrow \quad \frac{dx_1(t)}{dt} = \frac{de_o(t)}{dt} = x_2(t) \\ x_2(t) = \frac{de_o(t)}{dt} \\ u(t) = e_i(t) \\ y(t) = e_o(t) \quad \Rightarrow \quad y(t) = x_1(t) \end{gathered} \tag{1.25}$$

and combining it with (1.24) we get

$$\frac{dx_2(t)}{dt} + \left(\frac{L + R_1R_2C}{R_2LC}\right)x_2(t) + \left(\frac{R_1 + R_2}{R_2LC}\right)x_1(t) = \frac{1}{LC}u(t) \tag{1.26}$$

The first equation in (1.25) and equation (1.26) can be put into matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{R_1+R_2}{R_2LC} & -\frac{L+R_1R_2C}{R_2LC} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u \tag{1.27}$$

The last equation from (1.25), in matrix form, is written as

$$y(t) = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{1.28}$$

Equations (1.27) and (1.28) represent the state space form for the system whose mathematical model is given by (1.24). The corresponding state space matrices for this system are given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{R_1+R_2}{R_2LC} & -\frac{L+R_1R_2C}{R_2LC} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0], \quad \mathbf{D} = 0 \tag{1.29}$$

The state space form of a system is not unique. Using another change of variables, we can get, for the same system, another state space form, which is demonstrated in Problem 1.1.

The transfer function of this single-input single-output system is easily obtained by taking the Laplace transform of (1.24), which leads to

$$G(s) = \frac{\mathcal{L}\{e_0(t)\}}{\mathcal{L}\{e_i(t)\}} = \frac{\frac{1}{LC}}{s^2 + \frac{L+R_1R_2C}{R_2LC}s + \frac{R_1+R_2}{R_2LC}} \tag{1.30}$$

Note that a systematic approach for getting the state space form from differential (difference) equations will be given in detail in Chapter 3. In this chapter we present only the simplest cases. These cases are dictated by system physical structures described by a set of first- and second-order differential (difference) equations, which can be put in a straightforward way into matrix form, which in fact represents the desired state space form.

Another example, which demonstrates how to get a mathematical model for a real physical system, is taken from mechanical engineering.

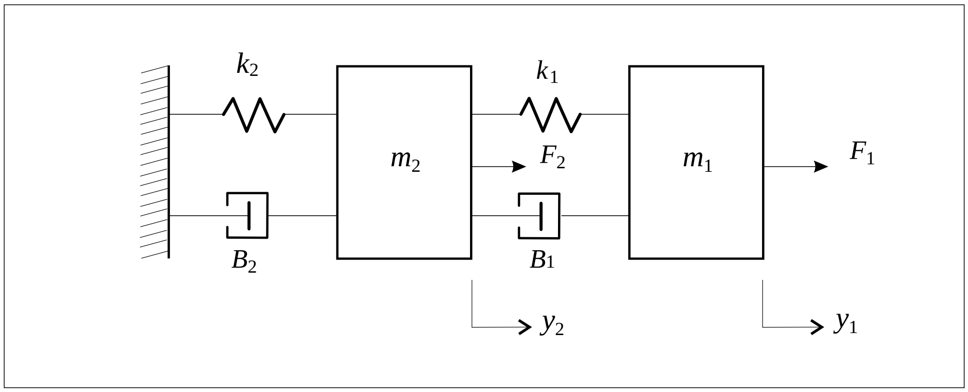

Figure 1.5: A translational mechanical system

A translational mechanical system is represented in Figure 1.5. The following two equations of motion for this system can be written by using the basic laws of dynamics (Greenwood, 1988)

$$F_1 = m_1 \frac{d^2 y_1}{dt^2} + B_1 \left(\frac{dy_1}{dt} - \frac{dy_2}{dt} \right) + k_1 (y_1 - y_2) \tag{1.31}$$

and

$$F_2 = m_2 \frac{d^2 y_2}{dt^2} + B_2 \frac{dy_2}{dt} + k_2 y_2 - B_1 \left(\frac{dy_1}{dt} - \frac{dy_2}{dt} \right) - k_1 (y_1 - y_2) \tag{1.32}$$

It can be seen that this system has two inputs, F_1 and F_2, and two outputs, y_1 and y_2. The rearranged form of equations (1.31) and (1.32) is given by

$$m_1 \frac{d^2 y_1}{dt^2} + B_1 \frac{dy_1}{dt} + k_1 y_1 - B_1 \frac{dy_2}{dt} - k_1 y_2 = F_1 \tag{1.33}$$

and

$$-B_1 \frac{dy_1}{dt} - k_1 y_1 + m_2 \frac{d^2 y_2}{dt^2} + (B_1 + B_2) \frac{dy_2}{dt} + (k_1 + k_2) y_2 = F_2 \tag{1.34}$$

From equations (1.33) and (1.34) the state space form can be obtained easily by choosing the following state space variables

$$\begin{gathered} x_1 = y_1, \quad x_2 = \frac{dy_1}{dt}, \quad x_3 = y_2, \quad x_4 = \frac{dy_2}{dt} \\ u_1 = F_1, \quad u_2 = F_2 \end{gathered} \tag{1.35}$$

The state space form of this two-input two-output system is given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -\frac{k_1}{m_1} & -\frac{B_1}{m_1} & \frac{k_1}{m_1} & \frac{B_1}{m_1}\\ 0 & 0 & 0 & 1\\ \frac{k_1}{m_2} & \frac{B_1}{m_2} & -\frac{k_1+k_2}{m_2} & -\frac{B_1+B_2}{m_2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0 & 0\\ \frac{1}{m_1} & 0\\ 0 & 0\\ 0 & \frac{1}{m_2}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} \tag{1.36}$$

and

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} \tag{1.37}$$

It is interesting to find the transfer function for this multi-input multi-output system. Taking the Laplace transforms of (1.33) and (1.34), and assuming that all initial conditions are equal to zero, we get the scalar transfer functions from each input to each output. This is obtained by keeping the input under consideration different from zero and setting the other one to zero, that is

$$\begin{aligned}
G_{11}(s) &= \left(\frac{Y_1(s)}{U_1(s)}\right)_{|U_2(s)=0} = \frac{m_2 s^2 + (B_1+B_2)s + (k_1+k_2)}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}\\
G_{12}(s) &= \left(\frac{Y_1(s)}{U_2(s)}\right)_{|U_1(s)=0} = \frac{B_1 s + k_1}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}\\
G_{21}(s) &= \left(\frac{Y_2(s)}{U_1(s)}\right)_{|U_2(s)=0} = \frac{B_1 s + k_1}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}\\
G_{22}(s) &= \left(\frac{Y_2(s)}{U_2(s)}\right)_{|U_1(s)=0} = \frac{m_1 s^2 + B_1 s + k_1}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}
\end{aligned} \tag{1.38}$$

where

$$\begin{aligned}
a_4 &= m_1 m_2\\
a_3 &= B_1(m_1+m_2) + m_1 B_2\\
a_2 &= B_1 B_2 + k_1(m_1+m_2) + k_2 m_1\\
a_1 &= k_1 B_2 + k_2 B_1\\
a_0 &= k_1 k_2
\end{aligned}$$

so that the system transfer function is given by

$$\mathbf{G}(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \tag{1.39}$$

Sometimes due to the complexity of dynamic systems it is not possible to establish mathematical relations describing the dynamical behavior of the systems under consideration. In those cases one has to use experimentation in order to get data that can be used in establishing some mathematical relations caused (induced) by system dynamics. The experimental way of getting system models is the subject of the area of control systems known as *system identification*. More about system identification can be found in Chapter 10. A classic textbook on system identification is given in the list of references (Ljung, 1987).

The reader particularly interested in mathematical modelling of real physical systems is referred to Wellstead (1979) and Kecman (1988).

1.5 Models of Some Control Systems

In this section we consider the modeling of two common physical control systems. In that direction modeling of an inverted pendulum and a complex electrical network is presented. Mathematical models of a DC motor will be derived in Section 2.2.1. DC motors are integral parts of several control system schemes. Mathematical models of many other interesting control systems can be found in Wellstead (1979), Kecman (1988), and Dorf (1992).

Inverted Pendulum

The inverted pendulum is a familiar dynamic system used very often in textbooks (Kwakernaak and Sivan, 1972; Kamen, 1990). Here, we follow derivations of Kecman (1988) for an idealized pendulum of length l whose mass, m_1, is concentrated at its end. It is assumed that a cart of mass m_2 is subjected to an external force F, which, as a control variable, has to keep the pendulum upright. Cart displacement is denoted by x and pendulum displacement is represented by angle θ (see Figure 1.6).

Using elementary knowledge from college physics (see for example Serway 1992), the equation of motion for translation in the direction of x axis is obtained by applying Newton's law

$$m_2 \frac{d^2 x(t)}{dt^2} + m_1 \frac{d^2 (x + l \sin\theta)}{dt^2} = F \tag{1.40}$$

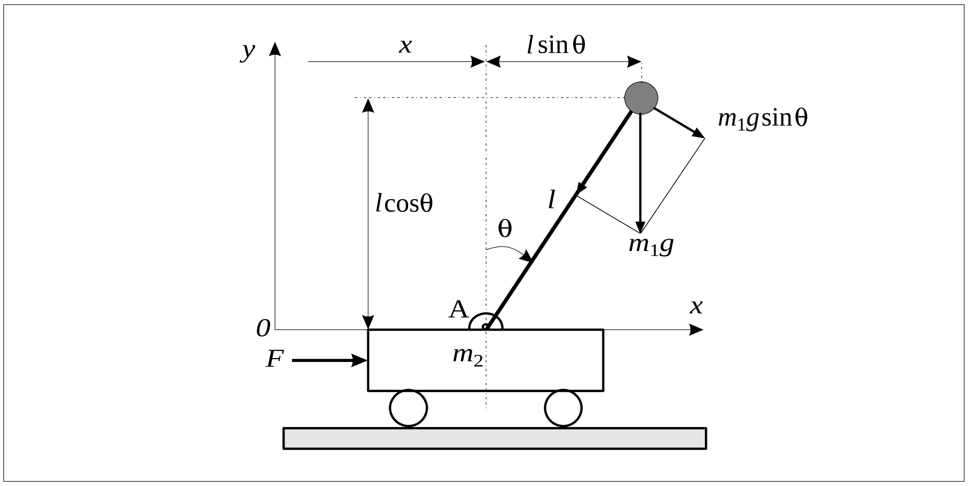

Figure 1.6: Inverted pendulum

The conservation of momentum equation with respect to rotation about point A implies

$$m_1 l \cos\theta \frac{d^2}{dt^2}(x + l\sin\theta) - m_1 l \sin\theta \frac{d^2}{dt^2}(l\cos\theta) = m_1 g l \sin\theta \tag{1.41}$$

where $g = 9.8\,\mathrm{m/s^2}$ is the gravitational constant. Bearing in mind that

$$\begin{gathered}
\frac{d}{dt}\sin\theta = \cos\theta\frac{d\theta}{dt}, \qquad \frac{d}{dt}\cos\theta = -\sin\theta\frac{d\theta}{dt} \\
\frac{d^2}{dt^2}\sin\theta = -\sin\theta\left(\frac{d\theta}{dt}\right)^2 + \cos\theta\frac{d^2\theta}{dt^2} \\
\frac{d^2}{dt^2}\cos\theta = -\cos\theta\left(\frac{d\theta}{dt}\right)^2 - \sin\theta\frac{d^2\theta}{dt^2}
\end{gathered} \tag{1.42}$$

we get a system of two second-order differential equations

$$\begin{gathered}
(m_1 + m_2)\frac{d^2x}{dt^2} - m_1 l \sin\theta\left(\frac{d\theta}{dt}\right)^2 + m_1 l \cos\theta\frac{d^2\theta}{dt^2} = F \\
\cos\theta\frac{d^2x}{dt^2} + l\frac{d^2\theta}{dt^2} = g\sin\theta
\end{gathered} \tag{1.43}$$

Equations (1.43) represent the desired mathematical model of an inverted pendulum.

Complex Electrical Network

Complex electrical networks are obtained by connecting basic electrical elements: resistors, inductors, capacitors (passive elements) and voltage and current sources (active elements). Of course, other electrical engineering elements like diodes and transistors can be present, but since in this course we study only the linear time invariant networks and since this textbook is intended for all engineering students we will limit our attention to basic electrical elements. The complexity of the network will be manifested by a large number of passive and active elements and large number of loops. Such a network is represented in Figure 1.7.

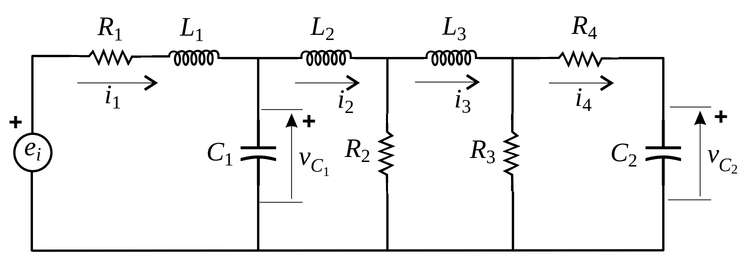

Figure 1.7: Complex electrical network

In electrical networks composed of inductors and capacitors *the total number of inductors and capacitors indicates the order of the dynamic system*. In this particular example, we have three inductors and two capacitors, i.e. the order of this dynamical system is $n = 5$. Having a dynamic system of order $n = 5$ indicates that the required number of first-order differential equations to be set is five. If one sets more than n differential equations, for a system of order n, some of them are redundant. Redundant equations have to be eliminated since they do not carry any new information; they are just linear combinations of the remaining equations. Using basic laws for currents and voltages, we can easily

set up five first-order differential equations; for voltages around loops

$$\begin{aligned} e_i - L_1\frac{di_1}{dt} - R_1 i_1 - v_{c_1} &= 0 \\ L_2\frac{di_2}{dt} - v_{c_1} + R_2(i_2 - i_3) &= 0 \\ L_3\frac{di_3}{dt} + R_3(i_3 - i_4) - R_2(i_2 - i_3) &= 0 \end{aligned} \tag{1.44}$$

and for currents

$$\begin{aligned} i_1 - i_2 - C_1\frac{dv_{c_1}}{dt} &= 0 \\ i_4 - C_2\frac{dv_{c_2}}{dt} &= 0 \end{aligned} \tag{1.45}$$

We have set up five equations for four currents and two voltages. Current i_4 can be eliminated by using the following algebraic voltage balance equation, which is valid for the last loop

$$R_3(i_3 - i_4) = R_4 i_4 + v_{c_2} \tag{1.46}$$

from which the current i_4 can be expressed as

$$i_4 = \frac{R_3}{R_3 + R_4}i_3 - \frac{1}{R_3 + R_4}v_{c_2} \tag{1.47}$$

Replacing current i_4 in (1.44) and (1.45) by the expression obtained in (1.47), the following five first-order differential equations are obtained

$$\begin{aligned} \frac{di_1}{dt} &= -\frac{R_1}{L_1}i_1 - \frac{1}{L_1}v_{c_1} + \frac{1}{L_1}e_i \\ \frac{di_2}{dt} &= -\frac{R_2}{L_2}i_2 + \frac{R_2}{L_3}i_3 + \frac{1}{L_2}v_{c_1} \\ \frac{di_3}{dt} &= \frac{R_2}{L_3}i_2 - \frac{R_2R_3 + R_2R_4 + R_3R_4}{L_3(R_3 + R_4)}i_3 - \frac{R_3}{L_3(R_3 + R_4)}v_{c_2} \\ \frac{dv_{c_1}}{dt} &= \frac{1}{C_1}i_1 - \frac{1}{C_1}i_2 \\ \frac{dv_{c_2}}{dt} &= \frac{R_3}{C_2(R_3 + R_4)}i_3 - \frac{1}{C_2(R_3 + R_4)}v_{c_2} \end{aligned} \tag{1.48}$$

The matrix form of this system of first-order differential equations represents the system state space form. Take $x_1 = i_1, x_2 = i_2, x_3 = i_3, x_4 = v_{c_1}, x_5 = v_{c_2}$, $u = e_i$, then the state space form is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & 0 & -\frac{1}{L_1} & 0 \\ 0 & -\frac{R_2}{L_2} & \frac{R_2}{L_2} & \frac{1}{L_2} & 0 \\ 0 & \frac{R_2}{L_3} & a_{33} & 0 & -\frac{R_3}{L_3(R_3+R_4)} \\ \frac{1}{C_1} & -\frac{1}{C_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{R_3}{C_2(R_3+R_4)} & 0 & -\frac{1}{C_2(R_3+R_4)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \tag{1.49}$$

where $a_{33} = -\frac{R_2R_3+R_2R_4+R_3R_4}{L_3(R_3+R_4)}$.

Note that while modeling electrical networks, it is advisable to use the currents through inductors and the voltages on capacitors for the state space variables, which, in fact, is done in this example.

We would like to point out that in this chapter we have presented only modeling and mathematical models for continuous-time real dynamic physical systems. The reason for this is twofold: (1) there are no real-world discrete-time physical dynamic systems; (2) discrete-time models obtained from social, economic, hydrological, and meteorological sciences are usually of no interest to control engineers. However, discrete-time models obtained by discretization of continuous-time systems will be treated in detail in Chapter 3.

1.6 Linearization of Nonlinear Systems

We have mentioned before that in this book we study only time invariant linear control systems and that the study of nonlinear control systems is rather difficult. However, in some cases it is possible to linearize nonlinear control systems and study them as linear ones. In this section we show how to perform linearization of control systems described by nonlinear differential equations. The procedure introduced is based on the Taylor series expansion and on the knowledge of nominal (operating) system trajectories and nominal control inputs. Readers particularly interested in the study of nonlinear systems are referred to a comprehensive book by Khalil (1992).

We will start with a simple scalar first-order nonlinear dynamic system represented by

$$\dot{x}(t) = f(x(t), u(t)) \tag{1.50}$$

Assume that under usual working circumstances this system operates along the trajectory $x_n(t)$ while it is driven by the control input $u_n(t)$. We call $x_n(t)$ and $u_n(t)$, respectively, the *nominal system trajectory* and the *nominal control input*. On the nominal trajectory the following differential equation is satisfied

$$\dot{x}_n(t) = f(x_n(t), u_n(t)) \tag{1.51}$$

Now assume that the motion of the nonlinear system (1.50) is in the neighborhood of the nominal system trajectory and that the distance from the nominal trajectory is small, that is

$$x(t) = x_n(t) + \Delta x(t) \tag{1.52}$$

where $\Delta x(t)$ represents a small quantity. It is natural to assume that the system motion in close proximity to the nominal trajectory will be sustained by a control input which is obtained by adding a small quantity to the nominal control input, that is

$$u(t) = u_n(t) + \Delta u(t) \tag{1.53}$$

For the system motion in close proximity to the nominal trajectory, from equations (1.50), (1.52), and (1.53), we have

$$\dot{x}_n(t) + \Delta\dot{x}(t) = f(x_n(t) + \Delta x(t), u_n(t) + \Delta u(t)) \tag{1.54}$$

Since $\Delta x(t)$ and $\Delta u(t)$ are small quantities, the right-hand side of (1.54) can be expanded into a Taylor series about the nominal trajectory and control, which produces

$$\begin{aligned}\dot{x}_n(t) + \Delta\dot{x}(t) = f(x_n(t), u_n(t)) + \frac{\partial f}{\partial x}(x_n, u_n)\Delta x(t) + \frac{\partial f}{\partial u}(x_n, u_n)\Delta u(t)\\ + \text{ higher–order terms}\end{aligned} \tag{1.55}$$

Using (1.51) and canceling higher-order terms (which contain very small quantities $\Delta x^2, \Delta u^2, \Delta x \Delta u, \Delta x^3, \ldots$), the following linear differential equation is obtained

$$\Delta \dot{x}(t) = \frac{\partial f}{\partial x}(x_n, u_n)\Delta x(t) + \frac{\partial f}{\partial u}(x_n, u_n)\Delta u(t) \tag{1.56}$$

whose solution represents a valid approximation for $\Delta x(t)$. Note that the *partial derivatives in the linearization procedure are evaluated at the nominal points*. Introducing the notation

$$a_0(t) = -\frac{\partial f}{\partial x}(x_n, u_n), \qquad b_0 = \frac{\partial f}{\partial u}(x_n, u_n) \tag{1.57}$$

the linear system (1.56) can be represented as

$$\Delta \dot{x}(t) + a_0(t)\Delta x(t) = b_0(t)\Delta u(t) \tag{1.58}$$

In general, the obtained linear system is time varying. Since in this course we study only time invariant systems, we will consider only those examples for which the linearization procedure produces time invariant systems. It remains to find the initial condition for the linearized system, which can be obtained from (1.52) as

$$\Delta x(t_0) = x(t_0) - x_n(t_0) \tag{1.59}$$

Similarly, we can linearize the second-order nonlinear dynamic system

$$\ddot{x} = f(x, \dot{x}, u, \dot{u}) \tag{1.60}$$

by assuming that

$$\begin{aligned} x(t) &= x_n(t) + \Delta x(t), \quad \dot{x}(t) = \dot{x}_n(t) + \Delta \dot{x}(t) \\ u(t) &= u_n(t) + \Delta u(t), \quad \dot{u}(t) = \dot{u}_n(t) + \Delta \dot{u}(t) \end{aligned} \tag{1.61}$$

and expanding

$$\ddot{x}_n + \Delta \ddot{x} = f(x_n + \Delta x_n, \dot{x}_n + \Delta \dot{x}, u_n + \Delta u, \dot{u}_n + \Delta \dot{u}) \tag{1.62}$$

into a Taylor series about nominal points $x_n, \dot{x}_n, u_n, \dot{u}_n$, which leads to

$$\Delta \ddot{x}(t) + a_1 \Delta \dot{x}(t) + a_0 \Delta x(t) = b_1 \Delta \dot{u}(t) + b_0 \Delta u(t) \tag{1.63}$$

where the corresponding coefficients are evaluated at the nominal points as

$$
\begin{aligned}
a_1 &= -\frac{\partial f}{\partial \dot{x}}(x_n, \dot{x}_n, u_n, \dot{u}_n), \quad a_0 = -\frac{\partial f}{\partial x}(x_n, \dot{x}_n, u_n, \dot{u}_n) \\
b_1 &= \frac{\partial f}{\partial \dot{u}}(x_n, \dot{x}_n, u_n, \dot{u}_n), \quad b_0 = \frac{\partial f}{\partial u}(x_n, \dot{x}_n, u_n, \dot{u}_n)
\end{aligned}
\tag{1.64}
$$

The initial conditions for the second-order linearized system are easily obtained from (1.61)

$$
\Delta x(t_0) = x(t_0) - x_n(t_0), \quad \Delta \dot{x}(t_0) = \dot{x}(t_0) - \dot{x}_n(t_0) \tag{1.65}
$$

Example 1.1: The mathematical model of a stick-balancing problem is given in Sontag (1990) by

$$
\ddot{\theta} = \sin\theta - u\cos\theta = f(\theta, u)
$$

where u is the horizontal force of a finger and θ represents the stick's angular displacement from the vertical. This second-order dynamic system is linearized at the nominal points ($\dot{\theta}_n(t) = \theta_n(t) = 0, u_n(t) = 0$) by using formulas (1.64), which produces

$$
a_1 = -\frac{\partial f}{\partial \dot{\theta}} = 0, \qquad a_0 = -\left(\frac{\partial f}{\partial \theta}\right)_{|_n} = -(\cos\theta + u\sin\theta)_{|_{\substack{\theta_n(t)=0 \\ u_n(t)=0}}} = -1
$$

$$
b_1 = \frac{\partial f}{\partial \dot{u}} = 0, \qquad b_0 = \left(\frac{\partial f}{\partial u}\right)_{|_n} = -(\cos\theta)_{|_{\theta_n(t)=0}} = -1
$$

The linearized equation is given by

$$
\ddot{\theta}(t) - \theta(t) = -u(t)
$$

Note that $\Delta\theta(t) = \theta(t), \Delta u(t) = u(t)$ since $\theta_n(t) = 0, u_n(t) = 0$. It is important to point out that the same linearized model could have been obtained by setting $\sin\theta(t) \approx \theta(t)$, $\cos\theta(t) \approx 1$, which is valid for small values of $\theta(t)$.

◇

Of course, we can extend the presented linearization procedure to an n-order nonlinear dynamic system with one input and one output in a straightforward

way. However, for multi-input multi-output systems this procedure becomes cumbersome. Using the state space model, the linearization procedure for the multi-input multi-output case is quite simple.

Consider now the general nonlinear dynamic control system in matrix form represented by

$$\frac{d}{dt}\mathbf{x}(t) = \mathcal{F}(\mathbf{x}(t), \mathbf{u}(t)) \tag{1.66}$$

where $\mathbf{x}(t)$, $\mathbf{u}(t)$, and $\mathcal{F}$ are, respectively, the n-dimensional state space vector, the r-dimensional control vector, and the n-dimensional vector function. Assume that the nominal (operating) system trajectory $\mathbf{x}_n(t)$ is known and that the nominal control that keeps the system on the nominal trajectory is given by $\mathbf{u}_n(t)$. Using the same logic as for the scalar case, we can assume that the actual system dynamics in the immediate proximity of the system nominal trajectories can be approximated by the first terms of the Taylor series. That is, starting with

$$\mathbf{x}(t) = \mathbf{x}_n(t) + \Delta\mathbf{x}(t), \qquad \mathbf{u}(t) = \mathbf{u}_n(t) + \Delta\mathbf{u}(t) \tag{1.67}$$

and

$$\frac{d}{dt}\mathbf{x}_n(t) = \mathcal{F}(\mathbf{x}_n(t), \mathbf{u}_n(t)) \tag{1.68}$$

we expand equation (1.66) as follows

$$\begin{aligned} \frac{d}{dt}\mathbf{x}_n + \frac{d}{dt}\Delta\mathbf{x} &= \mathcal{F}(\mathbf{x}_n + \Delta\mathbf{x}, \mathbf{u}_n + \Delta\mathbf{u}) \\ = \mathcal{F}(\mathbf{x}_n, \mathbf{u}_n) + \left(\frac{\partial\mathcal{F}}{\partial\mathbf{x}}\right)_{\left|\substack{\mathbf{x}_n(t) \\ \mathbf{u}_n(t)}\right.} &\Delta\mathbf{x} + \left(\frac{\partial\mathcal{F}}{\partial\mathbf{u}}\right)_{\left|\substack{\mathbf{x}_n(t) \\ \mathbf{u}_n(t)}\right.} \Delta\mathbf{u} \ + \ \text{higher--order terms} \end{aligned} \tag{1.69}$$

Higher-order terms contain at least quadratic quantities of $\Delta\mathbf{x}$ and $\Delta\mathbf{u}$. Since $\Delta\mathbf{x}$ and $\Delta\mathbf{u}$ are small their squares are even smaller, and hence the higher-order terms can be neglected. Using (1.67) and neglecting higher-order terms, an approximation is obtained

$$\frac{d}{dt}\Delta\mathbf{x}(t) = \left(\frac{\partial\mathcal{F}}{\partial\mathbf{x}}\right)_{\left|\substack{\mathbf{x}_n(t) \\ \mathbf{u}_n(t)}\right.} \Delta\mathbf{x}(t) + \left(\frac{\partial\mathcal{F}}{\partial\mathbf{u}}\right)_{\left|\substack{\mathbf{x}_n(t) \\ \mathbf{u}_n(t)}\right.} \Delta\mathbf{u}(t) \tag{1.70}$$

Partial derivatives in (1.70) represent the Jacobian matrices given by

$$\left(\frac{\partial \mathcal{F}}{\partial \mathbf{x}}\right)_{|\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} = \mathbf{A}^{n\times n} = \begin{bmatrix} \frac{\partial \mathcal{F}_1}{\partial x_1} & \frac{\partial \mathcal{F}_1}{\partial x_2} & \cdots & \cdots & \frac{\partial \mathcal{F}_1}{\partial x_n} \\ \frac{\partial \mathcal{F}_2}{\partial x_1} & \cdots & \cdots & \cdots & \frac{\partial \mathcal{F}_2}{\partial x_n} \\ \cdots & \cdots & \frac{\partial \mathcal{F}_i}{\partial x_j} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathcal{F}_n}{\partial x_1} & \frac{\partial \mathcal{F}_n}{\partial x_2} & \cdots & \cdots & \frac{\partial \mathcal{F}_n}{\partial x_n} \end{bmatrix}_{|\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} \tag{1.71a}$$

$$\left(\frac{\partial \mathcal{F}}{\partial \mathbf{u}}\right)_{|\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} = \mathbf{B}^{n\times r} = \begin{bmatrix} \frac{\partial \mathcal{F}_1}{\partial u_1} & \frac{\partial \mathcal{F}_1}{\partial u_2} & \cdots & \cdots & \frac{\partial \mathcal{F}_1}{\partial u_r} \\ \frac{\partial \mathcal{F}_2}{\partial u_1} & \cdots & \cdots & \cdots & \frac{\partial \mathcal{F}_2}{\partial u_r} \\ \cdots & \cdots & \frac{\partial \mathcal{F}_i}{\partial u_j} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathcal{F}_n}{\partial u_1} & \frac{\partial \mathcal{F}_n}{\partial u_2} & \cdots & \cdots & \frac{\partial \mathcal{F}_n}{\partial u_r} \end{bmatrix}_{|\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} \tag{1.71b}$$

Note that the Jacobian matrices have to be evaluated at the nominal points, i.e. at $\mathbf{x}_n(t)$ and $\mathbf{u}_n(t)$. With this notation, the linearized system (1.70) has the form

$$\frac{d}{dt}\Delta\mathbf{x}(t) = \mathbf{A}\Delta\mathbf{x}(t) + \mathbf{B}\Delta\mathbf{u}(t), \quad \Delta\mathbf{x}(t_0) = \mathbf{x}(t_0) - \mathbf{x}_n(t_0) \tag{1.72}$$

The output of a nonlinear system, in general, satisfies a nonlinear algebraic equation, that is

$$\mathbf{y}(t) = \mathcal{G}(\mathbf{x}(t), \mathbf{u}(t)) \tag{1.73}$$

This equation can be also linearized by expanding its right-hand side into a Taylor series about nominal points $\mathbf{x}_n(t)$ and $\mathbf{u}_n(t)$. This leads to

$$\mathbf{y}_n + \Delta\mathbf{y} = \mathcal{G}(\mathbf{x}_n, \mathbf{u}_n) + \left(\frac{\partial \mathcal{G}}{\partial \mathbf{x}}\right)_{|\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} \Delta\mathbf{x} + \left(\frac{\partial \mathcal{G}}{\partial \mathbf{u}}\right)_{|\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} \Delta\mathbf{u} \\ + \text{ higher–order terms} \tag{1.74}$$

Note that $\mathbf{y}_n$ cancels term $\mathcal{G}(\mathbf{x}_n, \mathbf{u}_n)$. By neglecting higher-order terms in (1.74), the linearized part of the output equation is given by

$$\Delta\mathbf{y}(t) = \mathbf{C}\boldsymbol{\Delta}\mathbf{x}(t) + \mathbf{D}\boldsymbol{\Delta}\mathbf{u}(t) \tag{1.75}$$

where the Jacobian matrices $\mathbf{C}$ and $\mathbf{D}$ satisfy

$$\mathbf{C}^{p\times n} = \left(\frac{\partial \mathcal{G}}{\partial \mathbf{x}}\right)_{\Big|_{\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}}} = \begin{bmatrix} \frac{\partial \mathcal{G}_1}{\partial x_1} & \frac{\partial \mathcal{G}_1}{\partial x_2} & \cdots & \cdots & \frac{\partial \mathcal{G}_1}{\partial x_n} \\ \frac{\partial \mathcal{G}_2}{\partial x_1} & \cdots & \cdots & \cdots & \frac{\partial \mathcal{G}_2}{\partial x_n} \\ \cdots & \cdots & \frac{\partial \mathcal{G}_i}{\partial x_j} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathcal{G}_p}{\partial x_1} & \frac{\partial \mathcal{G}_p}{\partial x_2} & \cdots & \cdots & \frac{\partial \mathcal{G}_p}{\partial x_n} \end{bmatrix}_{\Big|_{\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}}} \tag{1.76a}$$

$$\mathbf{D}^{p\times r} = \left(\frac{\partial \mathcal{G}}{\partial \mathbf{u}}\right)_{\Big|_{\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}}} = \begin{bmatrix} \frac{\partial \mathcal{G}_1}{\partial u_1} & \frac{\partial \mathcal{G}_1}{\partial u_2} & \cdots & \cdots & \frac{\partial \mathcal{G}_1}{\partial u_r} \\ \frac{\partial \mathcal{G}_2}{\partial u_1} & \cdots & \cdots & \cdots & \frac{\partial \mathcal{G}_2}{\partial u_r} \\ \cdots & \cdots & \frac{\partial \mathcal{G}_i}{\partial u_j} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathcal{G}_p}{\partial u_1} & \frac{\partial \mathcal{G}_p}{\partial u_2} & \cdots & \cdots & \frac{\partial \mathcal{G}_p}{\partial u_r} \end{bmatrix}_{\Big|_{\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}}} \tag{1.76b}$$

Example 1.2: Let a nonlinear system be represented by

$$\frac{dx_1}{dt} = x_1 \sin x_2 + x_2 u$$

$$\frac{dx_2}{dt} = x_1 e^{-x_2} + u^2$$

$$y = 2x_1 x_2 + x_2^2$$

Assume that the values for the system nominal trajectories and control are known and given by x_{1n}, x_{2n}, and u_n. The linearized state space equation of the above nonlinear system is obtained as

$$\begin{bmatrix} \Delta \dot{x}_1(t) \\ \Delta \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \sin x_{2n} & x_{1n}\cos x_{2n} + u_n \\ e^{-x_{2n}} & -x_{1n}e^{-x_{2n}} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} + \begin{bmatrix} x_{2n} \\ 2u_n \end{bmatrix} \Delta u(t)$$

$$\Delta y(t) = \begin{bmatrix} 2x_{2n} & 2x_{1n} + 2x_{2n} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} + 0\Delta u(t)$$

Having obtained the solution of this linearized system under the given control input $\Delta u(t)$, the corresponding approximation of the nonlinear system trajectories is

$$\mathbf{x}_n(t) + \Delta \mathbf{x}(t) = \begin{bmatrix} x_{1n}(t) \\ x_{2n}(t) \end{bmatrix} + \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}$$

◇

Example 1.3: Consider the mathematical model of a single-link robotic manipulator with a flexible joint (Spong and Vidyasagar, 1989)

$$I\ddot{\theta}_1 + mgl\sin\theta_1 + k(\theta_1 - \theta_2) = 0$$
$$J\ddot{\theta}_2 - k(\theta_1 - \theta_2) = u$$

where θ_1, θ_2 are angular positions, I, J are moments of inertia, m and l are, respectively, the link's mass and length, and k is the link's spring constant. Introducing the change of variables as

$$x_1 = \theta_1,\ x_2 = \dot{\theta}_1,\ x_3 = \theta_2,\ x_4 = \dot{\theta}_2$$

the manipulator's state space nonlinear model equivalent to (1.66) is given by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{mgl}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3)\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= \frac{k}{J}(x_1 - x_3) + \frac{1}{J}u\end{aligned}$$

Take the nominal points as $(x_{1n}, x_{2n}, x_{3n}, x_{4n}, u_n)$, then the matrices $\mathbf{A}$ and $\mathbf{B}$ defined in (1.71) are given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0\\ -\frac{k+mgl\cos x_{1n}}{I} & 0 & \frac{k}{I} & 0\\ 0 & 0 & 0 & 1\\ \frac{k}{J} & 0 & -\frac{k}{J} & 0\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}0\\0\\0\\ \frac{1}{J}\end{bmatrix}$$

In Spong (1995) the following numerical values are used for system parameters: $mgl = 5,\ I = J = 1,\ k = 0.08$.

Assuming that the output variable is equal to the link's angular position, that is

$$y = x_1$$

the matrices $\mathbf{C}$ and $\mathbf{D}$, defined in (1.76), are given by

$$\mathbf{C} = [1 \quad 0 \quad 0 \quad 0], \quad \mathbf{D} = 0$$

◇

In the next example, we give state space matrices for two linearized models of an F-15 aircraft obtained by linearizing nonlinear equations for two sets of operating points.

Example 1.4: F-15 Aircraft

The longitudinal dynamics of an F-15 aircraft can be represented by a fourth-order mathematical model. For two operating conditions (subsonic and supersonic) two linear mathematical models have been derived (Brumbaugh, 1994; Schomig *et al.*, 1995). The corresponding state space models are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -0.00819 & -25.70839 & 0 & -32.17095 \\ -0.00019 & -1.27626 & 1.0000 & 0 \\ 0.00069 & 1.02176 & -2.40523 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$+ \begin{bmatrix} -6.80939 \\ -0.14968 \\ -14.06111 \\ 0 \end{bmatrix} u, \qquad \mathbf{y} = \mathbf{x}$$

for subsonic flight conditions, and

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -0.01172 & -95.91071 & 0 & -32.11294 \\ -0.00011 & -1.87942 & 1.0000 & 0 \\ 0.00056 & -3.61627 & -3.44478 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$+ \begin{bmatrix} -25.40405 \\ -0.22042 \\ -53.42460 \\ 0 \end{bmatrix} u, \qquad \mathbf{y} = \mathbf{x}$$

for supersonic flight conditions.

Model derivations are beyond the scope of this book. The state space variables represent: $x_1(t)$—velocity in feet per second, $x_2(t)$—angle of attack in radians, $x_3(t)$—pitch rate in radians per second, and $x_4(t)$—pitch attitude in radians. The control input $u(t)$ represents the elevator control in radians.

◇

Linearization of an Inverted Pendulum

Sometimes is not necessary to go through the entire linearization procedure. It is possible to simplify and linearize mathematical equations describing a given dynamic system by using simple mathematics. This will be demonstrated on an example of the inverted pendulum considered in Section 1.5. A linearized model of the inverted pendulum can be obtained from equations (1.43) by assuming that in the normal operating position (pendulum in an upright position) the pendulum displacement θ is very small so that the following approximations are valid $\sin\theta(t) \approx \theta(t)$, $\cos\theta(t) \approx 1$, $\theta(t)(d\theta(t)/dt)^2 \approx 0, \forall t$. Then, from (1.43), the linearized model of the inverted pendulum is obtained as

$$\begin{aligned} (m_1+m_2)\frac{d^2x}{dt^2} + m_1 l\frac{d^2\theta}{dt^2} &= F \\ \frac{d^2x}{dt^2} + l\frac{d^2\theta}{dt^2} &= g\theta \end{aligned} \tag{1.77}$$

This model can easily be put in the state space form equivalent to (1.14) by introducing the following change of variables

$$\begin{aligned} x_1 &= x \;\Rightarrow\; \dot{x}_1 = x_2 \\ x_2 &= \frac{dx}{dt} = \dot{x} \\ x_3 &= \theta \;\Rightarrow\; \dot{x}_3 = x_4 \\ x_4 &= \frac{d\theta}{dt} \\ u &= F \end{aligned} \tag{1.78}$$

With this change of variables equations (1.77) imply

$$\begin{aligned} \dot{x}_2 &= -\frac{m_1 g}{m_2}x_3 + \frac{1}{m_2}u \\ \dot{x}_4 &= \frac{(m_1+m_2)g}{m_2 l}x_3 - \frac{1}{m_2 l}u \end{aligned} \tag{1.79}$$

From (1.78) and (1.79) the state space form of the inverted pendulum is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{m_1 g}{m_2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(m_1+m_2)g}{m_2 l} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_2} \\ 0 \\ -\frac{1}{m_2 l} \end{bmatrix} u \tag{1.80}$$

The output equation can be chosen such that it gives information about the cart's horizontal displacement x_1 and the link's angular position x_3. In that case a possible choice for the output equation is

$$y = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} \tag{1.81}$$

The same state variables will appear directly on the output if the following output equations are used

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} \tag{1.82}$$

$$\mathbf{y} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} \tag{1.83}$$

The main difference between (1.81) and (1.82)–(1.83) is that in output equations (1.82) and (1.83) we have two channels, each of which independently produces information about the particular state variable.

Finally, we would like to point out that the SIMULINK package is very convenient for simulation of nonlinear systems. It can also be used to obtain linearized models of nonlinear systems around given operating points (nominal system trajectories and controls).

1.7 MATLAB Computer Analysis and Design

MATLAB is a very advanced and reliable computer package, which can be used for computer-aided control system analysis and design. In addition to handling standard linear algebra problems, it has several specialized control theory and application toolboxes. One of them, the CONTROL toolbox, will be extensively used in this book. At some places in the book we also refer to the SIMULINK package, which is very convenient for simulation (finding system responses due to given inputs) of linear and nonlinear systems. MATLAB is user friendly. It takes only a few hours to master all of its functions. MATLAB will help students obtain a deeper understanding of the main control theory concepts and techniques, and to study higher-order real physical control systems, which would be impossible using only pencil and paper. Many MATLAB problems, laboratory experiments, and case studies will be encountered in this book. More about MATLAB and its CONTROL toolbox can be found in Appendix D.

1.8 Book Organization

This book has two parts: Part I, titled *Methods and Concepts*, Chapters 2–5, and Part II, titled *Analysis and Design*, Chapters 6–9. In the first part of the book, in Chapters 2 and 3, two main techniques of control theory—the transfer function approach and the state space method—are discussed in detail. In Chapter 4 we consider the concepts of system controllability and observability, and in Chapter 5 the stability concept of time-invariant continuous and discrete systems is presented.

In this introductory chapter we have defined the general control problem. The main control system characteristics and control objectives will be presented in the second part of this book starting with Chapter 6. In Chapter 6, the control system specifications relating to a system's transient behavior and steady state properties will be considered. Since the emphasis in this book is on the time domain controller design (based on the root locus technique), the corresponding technique is presented in detail in Chapter 7. Design of controllers that solve specific control problems will be presented in Chapters 8 and 9. In Chapter 10 we give an overview of modern control theory, which can serve as an introduction for further studies of control theory and its applications. The presentation of Chapter 10 to undergraduate students can either be completely omitted or even expanded at schools that have strong programs in control systems.

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1.10 Problems

1.1 Find the state space form for the electrical circuit whose mathematical model is given in (1.20)–(1.22) by taking for the state space variables the input current and output voltage, i.e. by choosing $x_1 = i_1, x_2 = e_o$. In addition, take $y = x_2$ and $u = e_i$.

1.2 Find a mathematical model for the inverted pendulum, assuming that its mass is concentrated at the center of gravity, and linearize the obtained nonlinear system at $\dot{\theta}_n(t) = \theta_n(t) = \dot{x}_n(t) = x_n(t) = 0$ (Kwakernaak and Sivan, 1972; Lewis, 1992). Compare the linear model obtained with model (1.80) derived under the assumption that the pendulum mass is concentrated at its end.

1.3 Verify the expressions given in (1.38) for the transfer function of a two-input two-output translational mechanical system.

1.4 Find the mathematical model, transfer function, and state space form of the electrical network given in Figure 1.8.

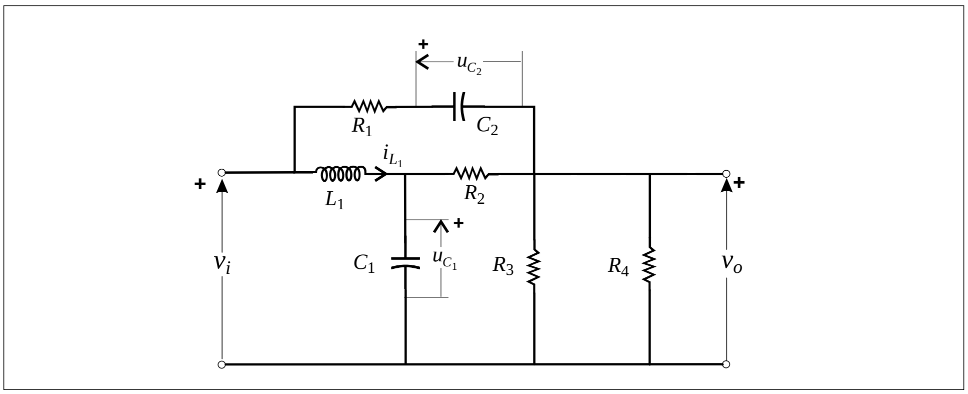

Figure 1.8: An electrical network

1.5 Linearize a scalar system represented by the first-order differential equation

$$\frac{dx(t)}{dt} = x(t)u(t)e^{-u(t)}, \qquad x(0) = 0.9$$

at a nominal point given by $(x_n(t), u_n(t)) = (1, 0)$.

1.6 Consider a nonlinear continuous-time system given by

$$\frac{d^2x(t)}{dt^2} = -2\frac{dx(t)}{dt}\cos u(t) - (1 + u(t))x(t) + 1, \quad x(0) = 1.1, \quad \frac{dx(0)}{dt} = 0.1$$

Derive its linearized equation with respect to a nominal point defined by $(x_n(t), u_n(t)) = (1, 0)$. Find the linearized system response due to $\Delta u(t) = e^{-2t}$.

1.7 For a nonlinear system

$$\frac{d^2x(t)}{dt^2}+2\frac{dx(t)}{dt}u(t)+(1-u(t))x(t)=u^2(t)+1, \quad x(0)=0,\ \frac{dx(0)}{dt}=1$$

find the nominal system response on the nominal system trajectory defined by $u_n(t) = 1$, subject to $x_n(0) = 0$ and $dx_n(0)/dt = 1.1$. Find the linearized state space equation and its initial conditions.

1.8 The mathematical model of a simple pendulum is given by (see for example Kamen, 1990)

$$I\frac{d^2\theta}{dt^2}+mgl\sin\theta = lu(t), \quad \theta(t_0)=\theta_0,\ \dot{\theta}(t_0)=\omega_0$$

where I is the moment of inertia, l, m are pendulum length and mass, respectively, and $u(t)$ is an external tangential force. Assume that $\theta_n(t) = 0$, $u_n(t) = 0, \theta_n(t_0) = 0, \dot{\theta}_n(t_0) = 0$, and θ_0, ω_0 are small. Find the linearized equation for this pendulum by using formulas (1.64). Determine the initial conditions.

1.9 Linearize the given system at a nominal point $(x_{1n}, x_{2n}, x_{3n}) = (0, 1, 1)$

$$\begin{aligned}\dot{x}_1 &= x_1x_2 - \sin x_1\\ \dot{x}_2 &= 1 - 3x_2e^{-x_1}\\ \dot{x}_3 &= x_1x_2x_3\end{aligned}$$

1.10 Linearize a nonlinear control system represented by

$$\begin{aligned}\dot{x}_1 &= u\ln x_1 + x_2e^{-u}\\ \dot{x}_2 &= x_1\sin u - \sin x_2\\ y &= \sin x_1\end{aligned}$$

Assume that x_{1n}, x_{2n}, and u_n are known. Find the transfer function of the linearized model obtained.

1.11 Linearize the Volterra predator–prey mathematical model

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1x_2\\ \dot{x}_2 &= x_2 - x_1x_2\end{aligned}$$

at a nominal point given by $(x_{1n}, x_{2n}) = (0, 0)$.

1.12 A linearized model of a single-link manipulator with a flexible joint is given by (Spong and Vidyasagar, 1989)

$$J_l\ddot{\theta}_l + B_l\dot{\theta}_l + k(\theta_l - \theta_m) = 0$$
$$J_m\ddot{\theta}_m + B_m\dot{\theta}_m - k(\theta_l - \theta_m) = u(t)$$

where J_l, J_m are moments of inertia, B_l, B_m are damping factors, k is the spring constant, $u(t)$ is the input torque, and $\theta_m(t), \theta_l(t)$ are angular positions. Write the state space form for this manipulator by taking the following change of variables: $x_1 = \theta_l, x_2 = \dot{\theta}_l, x_3 = \theta_m, x_4 = \dot{\theta}_m$.

Remark: Note that the SIMULINK package can be used for linearization of nonlinear systems. Students may check most of the linearization problems given in this section by using the `linmod` function of SIMULINK.